

Wiskunde: Calculus & Lineaire Algebra

Theorie

(2de Semester - Lineaire Algebra)

Lineaire Algebra

Stelsels Lineaire Vergelijkingen

Matrixrekening

Determinanten

Vectorruimten

Eigenwaarden en eigenvectoren

40% van het eindcijfer

20% StudyGPS

16% theoriegedeelte

64% oefeningen gedeelte

(moet meer dan 8/20 halen om door te zijn!)

Stelsels Lineaire Vergelijkingen

H1: Stelsysteem van lineaire vergelijkingen

A system of linear equations has

1. no solution, or
2. exactly one solution, or
3. infinitely many solutions.

A system of linear equations is said to be **consistent** if it has either one solution or infinitely many solutions; a system is **inconsistent** if it has no solution.

Het lineaire stelsel

$$\begin{aligned}x_1 - 2x_2 + x_3 &= 0 \\2x_2 - 8x_3 &= 8 \\-4x_1 + 5x_2 + 9x_3 &= -9\end{aligned}$$

heeft als uitgebreide matrix:

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

ELEMENTARY ROW OPERATIONS

1. (Replacement) Replace one row by the sum of itself and a multiple of another row.²
2. (Interchange) Interchange two rows.
3. (Scaling) Multiply all entries in a row by a nonzero constant.

If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

HL2: Rij reductie en echelonvorm

A rectangular matrix is in **echelon form** (or **row echelon form**) if it has the following three properties:

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form** (or **reduced row echelon form**):

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

Uniqueness of the Reduced Echelon Form

Each matrix is row equivalent to one and only one reduced echelon matrix.

A **pivot position** in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A . A **pivot column** is a column of A that contains a pivot position.

Existence and Uniqueness Theorem

A linear system is consistent if and only if the rightmost column of the augmented matrix is *not* a pivot column—that is, if and only if an echelon form of the augmented matrix has *no* row of the form

$$[0 \quad \cdots \quad 0 \quad b] \quad \text{with } b \text{ nonzero}$$

If a linear system is consistent, then the solution set contains either (i) a unique solution, when there are no free variables, or (ii) infinitely many solutions, when there is at least one free variable.

STEP 1

Begin with the leftmost nonzero column. This is a pivot column. The pivot position is at the top.

STEP 2

Select a nonzero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.

STEP 3

Use row replacement operations to create zeros in all positions below the pivot.

STEP 4

Cover (or ignore) the row containing the pivot position and cover all rows, if any, above it. Apply steps 1–3 to the submatrix that remains. Repeat the process until there are no more nonzero rows to modify.

USING ROW REDUCTION TO SOLVE A LINEAR SYSTEM

1. Write the augmented matrix of the system.
2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.
3. Continue row reduction to obtain the reduced echelon form.
4. Write the system of equations corresponding to the matrix obtained in step 3.
5. Rewrite each nonzero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.

H1.3: Vector vergelijkingen

Parallelogram Rule for Addition

If $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^2$ are represented as points in the plane, then $\mathbf{u} + \mathbf{v}$ corresponds to the fourth vertex of the parallelogram whose other vertices are $\mathbf{u}$, $\mathbf{0}$, and $\mathbf{v}$. See Fig. 3.

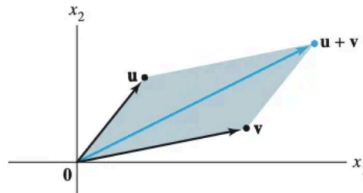


FIGURE 3 The parallelogram rule.

Algebraic Properties of $\mathbb{R}^n$

For all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $\mathbb{R}^n$ and all scalars c and d :

- | | |
|---|--|
| (i) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ | (v) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ |
| (ii) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ | (vi) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ |
| (iii) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$ | (vii) $c(d\mathbf{u}) = (cd)(\mathbf{u})$ |
| (iv) $\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = \mathbf{0}$,
where $-\mathbf{u}$ denotes $(-1)\mathbf{u}$ | (viii) $1\mathbf{u} = \mathbf{u}$ |

A vector equation

$$x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2 + \cdots + x_n \mathbf{a}_n = \mathbf{b}$$

has the same solution set as the linear system whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix} \quad (5)$$

In particular, $\mathbf{b}$ can be generated by a linear combination of $\mathbf{a}_1, \dots, \mathbf{a}_n$ if and only if there exists a solution to the linear system corresponding to the matrix (5).

If $\mathbf{v}_1, \dots, \mathbf{v}_p$ are in $\mathbb{R}^n$, then the set of all linear combinations of $\mathbf{v}_1, \dots, \mathbf{v}_p$ is denoted by $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ and is called the **subset of $\mathbb{R}^n$ spanned** (or **generated**) by $\mathbf{v}_1, \dots, \mathbf{v}_p$. That is, $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is the collection of all vectors that can be written in the form

$$c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \cdots + c_p \mathbf{v}_p$$

with $c_1, \dots, c_p$ scalars.

H1.4: De matrixvergelijking $A\vec{x} = \vec{b}$

If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \dots, \mathbf{a}_n$, and if $\mathbf{x}$ is in $\mathbb{R}^n$, then the **product of A and $\mathbf{x}$** , denoted by $A\mathbf{x}$, is **the linear combination of the columns of A using the corresponding entries in $\mathbf{x}$ as weights**; that is,

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$$

If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \dots, \mathbf{a}_n$, and if $\mathbf{b}$ is in $\mathbb{R}^m$, the matrix equation

$$A\mathbf{x} = \mathbf{b} \quad (4)$$

has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b} \quad (5)$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix} \quad (6)$$

The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a linear combination of the columns of A .

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular A , either they are all true statements or they are all false.

- For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.
- Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A .
- The columns of A span $\mathbb{R}^m$.
- A has a pivot position in every row.

Row-Vector Rule for Computing $A\mathbf{x}$

If the product $A\mathbf{x}$ is defined, then the i th entry in $A\mathbf{x}$ is the sum of the products of corresponding entries from row i of A and from the vector $\mathbf{x}$.

If A is an $m \times n$ matrix, $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, and c is a scalar, then:

- $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$;
- $A(c\mathbf{u}) = c(A\mathbf{u})$.

H1.5: Oplossingsverzamelingen van lineaire systemen

A system of linear equations is said to be **homogeneous** if it can be written in the form $A\mathbf{x} = \mathbf{0}$, where A is an $m \times n$ matrix and $\mathbf{0}$ is the zero vector in $\mathbb{R}^m$. Such a system $A\mathbf{x} = \mathbf{0}$ *always* has at least one solution, namely, $\mathbf{x} = \mathbf{0}$ (the zero vector in $\mathbb{R}^n$). This zero solution is usually called the **trivial solution**. For a given equation $A\mathbf{x} = \mathbf{0}$, the important question is whether there exists a **nontrivial solution**, that is, a nonzero vector $\mathbf{x}$ that satisfies $A\mathbf{x} = \mathbf{0}$. The Existence and Uniqueness Theorem in Section 1.2 (Theorem 2) leads immediately to the following fact.

The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if the equation has at least one free variable.

Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some given $\mathbf{b}$, and let $\mathbf{p}$ be a solution. Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, where $\mathbf{v}_h$ is any solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

WRITING A SOLUTION SET (OF A CONSISTENT SYSTEM) IN PARAMETRIC VECTOR FORM

1. Row reduce the augmented matrix to reduced echelon form.
2. Express each basic variable in terms of any free variables appearing in an equation.
3. Write a typical solution $\mathbf{x}$ as a vector whose entries depend on the free variables, if any.
4. Decompose $\mathbf{x}$ into a linear combination of vectors (with numeric entries) using the free variables as parameters.

HL.7: Lineair onafhankelijk

An indexed set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \dots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution. The set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exist weights $c_1, \dots, c_p$, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_p\mathbf{v}_p = \mathbf{0} \quad (2)$$

The columns of a matrix A are linearly independent if and only if the equation $A\mathbf{x} = \mathbf{0}$ has *only* the trivial solution. (3)

A set of two vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent if at least one of the vectors is a multiple of the other. The set is linearly independent if and only if neither of the vectors is a multiple of the other.

Characterization of Linearly Dependent Sets

An indexed set $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others. In fact, if S is linearly dependent and $\mathbf{v}_1 \neq \mathbf{0}$, then some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors, $\mathbf{v}_1, \dots, \mathbf{v}_{j-1}$.

If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.

If a set $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ contains the zero vector, then the set is linearly dependent.

H1.8: Inleiding tot lineaire transformaties

A transformation (or mapping) T is **linear** if:

- (i) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in the domain of T ;
- (ii) $T(c\mathbf{u}) = cT(\mathbf{u})$ for all scalars c and all $\mathbf{u}$ in the domain of T .

If T is a linear transformation, then

$$T(\mathbf{0}) = \mathbf{0} \quad (3)$$

and

$$T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v}) \quad (4)$$

for all vectors $\mathbf{u}, \mathbf{v}$ in the domain of T and all scalars c, d .

$$T(c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p) = c_1T(\mathbf{v}_1) + \cdots + c_pT(\mathbf{v}_p)$$

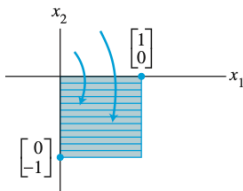
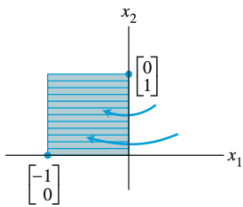
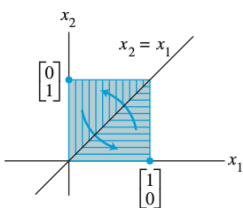
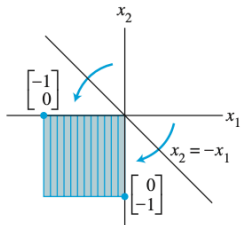
HL9: De matrix van een lineaire transformatie

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that

$$T(\mathbf{x}) = A\mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

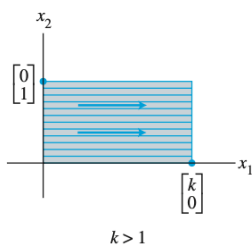
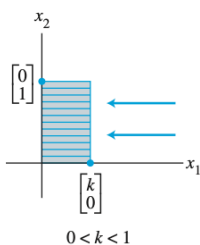
In fact, A is the $m \times n$ matrix whose j th column is the vector $T(\mathbf{e}_j)$, where $\mathbf{e}_j$ is the j th column of the identity matrix in $\mathbb{R}^n$:

$$A = [T(\mathbf{e}_1) \quad \cdots \quad T(\mathbf{e}_n)] \quad (3)$$

Transformation	Image of the Unit Square	Standard Matrix
Reflection through the x_1 -axis		$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Reflection through the x_2 -axis		$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
Reflection through the line $x_2 = x_1$		$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
Reflection through the line $x_2 = -x_1$		$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

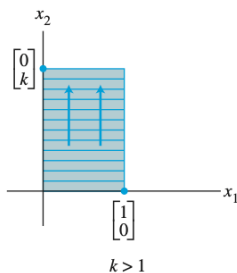
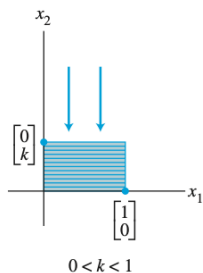
Transformation	Image of the Unit Square	Standard Matrix
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Horizontal contraction and expansion



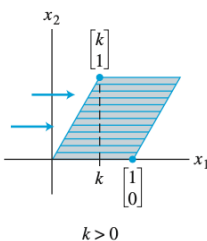
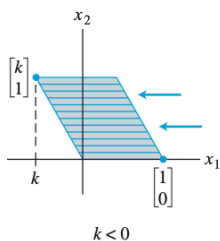
$$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$$

Vertical contraction and expansion



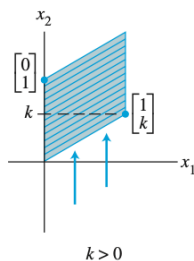
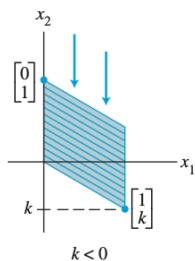
$$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$$

Horizontal shear



$$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

Vertical shear



$$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$$

Transformation	Image of the Unit Square	Standard Matrix
Projection onto the x_1 -axis		$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
Projection onto the x_2 -axis		$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$
Reflection through the origin		$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

A mapping $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be **onto** $\mathbb{R}^m$ if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of *at least one* $\mathbf{x}$ in $\mathbb{R}^n$.

A mapping $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be **one-to-one** if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of *at most one* $\mathbf{x}$ in $\mathbb{R}^n$.

onto = surjectief

one-to-one = injectief

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation and let A be the standard matrix for T . Then:

- a. T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$;
- b. T is one-to-one if and only if the columns of A are linearly independent.

Matrixrekening

HL: Matrixbewerkingen

If A is an $m \times n$ matrix—that is, a matrix with m rows and n columns—then the scalar entry in the i th row and j th column of A is denoted by a_{ij} and is called the (i, j) -entry of A . See Fig. 1. For instance, the $(3, 2)$ -entry is the number a_{32} in the third row, second column. Each column of A is a list of m real numbers, which identifies a vector in $\mathbb{R}^m$. Often, these columns are denoted by $\mathbf{a}_1, \dots, \mathbf{a}_n$, and the matrix A is written as

$$A = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n]$$

Observe that the number a_{ij} is the i th entry (from the top) of the j th column vector $\mathbf{a}_j$.

The **diagonal entries** in an $m \times n$ matrix $A = [a_{ij}]$ are $a_{11}, a_{22}, a_{33}, \dots$, and they form the **main diagonal** of A . A **diagonal matrix** is a square $n \times n$ matrix whose nondiagonal entries are zero. An example is the $n \times n$ identity matrix, I_n . An $m \times n$ matrix whose entries are all zero is a **zero matrix** and is written as 0 . The size of a zero matrix is usually clear from the context.

Let A , B , and C be matrices of the same size, and let r and s be scalars.

- | | |
|--------------------------------|-------------------------|
| a. $A + B = B + A$ | d. $r(A + B) = rA + rB$ |
| b. $(A + B) + C = A + (B + C)$ | e. $(r + s)A = rA + sA$ |
| c. $A + 0 = A$ | f. $r(sA) = (rs)A$ |

If A is an $m \times n$ matrix, and if B is an $n \times p$ matrix with columns $\mathbf{b}_1, \dots, \mathbf{b}_p$, then the product AB is the $m \times p$ matrix whose columns are $A\mathbf{b}_1, \dots, A\mathbf{b}_p$. That is,

$$AB = A[\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_p] = [A\mathbf{b}_1 \quad A\mathbf{b}_2 \quad \cdots \quad A\mathbf{b}_p]$$

Each column of AB is a linear combination of the columns of A using weights from the corresponding column of B .

ROW-COLUMN RULE FOR COMPUTING AB

If the product AB is defined, then the entry in row i and column j of AB is the sum of the products of corresponding entries from row i of A and column j of B . If $(AB)_{ij}$ denotes the (i, j) -entry in AB , and if A is an $m \times n$ matrix, then

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

Let A be an $m \times n$ matrix, and let B and C have sizes for which the indicated sums and products are defined.

- $A(BC) = (AB)C$ (associative law of multiplication)
- $A(B + C) = AB + AC$ (left distributive law)
- $(B + C)A = BA + CA$ (right distributive law)
- $r(AB) = (rA)B = A(rB)$
for any scalar r
- $I_m A = A = A I_n$ (identity for matrix multiplication)

WARNINGS:

- In general, $AB \neq BA$.
- The cancellation laws do *not* hold for matrix multiplication. That is, if $AB = AC$, then it is *not* true in general that $B = C$. (See Exercise 10.)
- If a product AB is the zero matrix, you *cannot* conclude in general that either $A = 0$ or $B = 0$. (See Exercise 12.)

Given an $m \times n$ matrix A , the **transpose** of A is the $n \times m$ matrix, denoted by A^T , whose columns are formed from the corresponding rows of A .

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}.$$

Let A and B denote matrices whose sizes are appropriate for the following sums and products.

- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$
- For any scalar r , $(rA)^T = rA^T$
- $(AB)^T = B^T A^T$

The transpose of a product of matrices equals the product of their transposes in the *reverse* order.

H2.2: De inverse van een matrix

$$A^{-1}A = I \quad \text{and} \quad AA^{-1} = I$$

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$, then A is not invertible.

$$\det A = ad - bc$$

If A is an invertible $n \times n$ matrix, then for each $\mathbf{b}$ in $\mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

a. If A is an invertible matrix, then A^{-1} is invertible and

$$(A^{-1})^{-1} = A$$

b. If A and B are $n \times n$ invertible matrices, then so is AB , and the inverse of AB is the product of the inverses of A and B in the reverse order. That is,

$$(AB)^{-1} = B^{-1}A^{-1}$$

c. If A is an invertible matrix, then so is A^T , and the inverse of A^T is the transpose of A^{-1} . That is,

$$(A^T)^{-1} = (A^{-1})^T$$

The product of $n \times n$ invertible matrices is invertible, and the inverse is the product of their inverses in the reverse order.

If an elementary row operation is performed on an $m \times n$ matrix A , the resulting matrix can be written as EA , where the $m \times m$ matrix E is created by performing the same row operation on I_m .

Each elementary matrix E is invertible. The inverse of E is the elementary matrix of the same type that transforms E back into I .

An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n , and in this case, any sequence of elementary row operations that reduces A to I_n also transforms I_n into A^{-1} .

ALGORITHM FOR FINDING A^{-1}

Row reduce the augmented matrix $[A \ I]$. If A is row equivalent to I , then $[A \ I]$ is row equivalent to $[I \ A^{-1}]$. Otherwise, A does not have an inverse.

H2.3: Eigenschappen van inverteerbare matrices

The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A , the statements are either all true or all false.

- A is an invertible matrix.
- A is row equivalent to the $n \times n$ identity matrix.
- A has n pivot positions.
- The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
- The columns of A form a linearly independent set.
- The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.
- The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.
- The columns of A span $\mathbb{R}^n$.
- The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.
- There is an $n \times n$ matrix C such that $CA = I$.
- There is an $n \times n$ matrix D such that $AD = I$.
- A^T is an invertible matrix.

Let A and B be square matrices. If $AB = I$, then A and B are both invertible, with $B = A^{-1}$ and $A = B^{-1}$.

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear transformation and let A be the standard matrix for T . Then T is invertible if and only if A is an invertible matrix. In that case, the linear transformation S given by $S(\mathbf{x}) = A^{-1}\mathbf{x}$ is the unique function satisfying equations (1) and (2).

H2.4: Blokmatrices (matrixpartities)

EXAMPLE 1 The matrix

$$A = \left[\begin{array}{ccc|cc|c} 3 & 0 & -1 & 5 & 9 & -2 \\ -5 & 2 & 4 & 0 & -3 & 1 \\ -8 & -6 & 3 & 1 & 7 & -4 \end{array} \right]$$

can also be written as the 2×3 **partitioned** (or **block**) **matrix**

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}$$

whose entries are the *blocks* (or *submatrices*)

$$A_{11} = \begin{bmatrix} 3 & 0 & -1 \\ -5 & 2 & 4 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 5 & 9 \\ 0 & -3 \end{bmatrix}, \quad A_{13} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} -8 & -6 & 3 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 1 & 7 \end{bmatrix}, \quad A_{23} = \begin{bmatrix} -4 \end{bmatrix} \quad \blacksquare$$

Addition and Scalar Multiplication

If matrices A and B are the same size and are partitioned in exactly the same way, then it is natural to make the same partition of the ordinary matrix sum $A + B$. In this case, each block of $A + B$ is the (matrix) sum of the corresponding blocks of A and B . Multiplication of a partitioned matrix by a scalar is also computed block by block.

Multiplication of Partitioned Matrices

Partitioned matrices can be multiplied by the usual row-column rule as if the block entries were scalars, provided that for a product AB , the column partition of A matches the row partition of B .

Column-Row Expansion of AB

If A is $m \times n$ and B is $n \times p$, then

$$\begin{aligned} AB &= [\text{col}_1(A) \quad \text{col}_2(A) \quad \cdots \quad \text{col}_n(A)] \begin{bmatrix} \text{row}_1(B) \\ \text{row}_2(B) \\ \vdots \\ \text{row}_n(B) \end{bmatrix} \\ &= \text{col}_1(A) \text{row}_1(B) + \cdots + \text{col}_n(A) \text{row}_n(B) \end{aligned} \quad (1)$$

H2.7: Toepassingen in grafische computerwetenschappen

Each point (x, y) in $\mathbb{R}^2$ can be identified with the point $(x, y, 1)$ on the plane in $\mathbb{R}^3$ that lies one unit above the xy -plane. We say that (x, y) has *homogeneous coordinates* $(x, y, 1)$. For instance, the point $(0, 0)$ has homogeneous coordinates $(0, 0, 1)$. Homogeneous coordinates for points are not added or multiplied by scalars, but they can be transformed via multiplication by 3×3 matrices.

EXAMPLE 4 A translation of the form $(x, y) \mapsto (x + h, y + k)$ is written in homogeneous coordinates as $(x, y, 1) \mapsto (x + h, y + k, 1)$. This transformation can be computed via matrix multiplication:

$$\begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + h \\ y + k \\ 1 \end{bmatrix} \quad \blacksquare$$

EXAMPLE 5 Any linear transformation on $\mathbb{R}^2$ is represented with respect to homogeneous coordinates by a partitioned matrix of the form $\begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}$, where A is a 2×2 matrix. Typical examples are

$$\begin{bmatrix} \cos \varphi & -\sin \varphi & 0 \\ \sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} s & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Counterclockwise rotation about the origin, angle φ Reflection through $y = x$ Scale x by s and y by t ■

Composite Transformations

The movement of a figure on a computer screen often requires two or more basic transformations. The composition of such transformations corresponds to matrix multiplication when homogeneous coordinates are used.

Determinanten

H3.1: Inleiding tot determinanten

For $n \geq 2$, the **determinant** of an $n \times n$ matrix $A = [a_{ij}]$ is the sum of n terms of the form $\pm a_{1j} \det A_{1j}$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \dots, a_{1n}$ are from the first row of A . In symbols,

$$\begin{aligned}\det A &= a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n} \\ &= \sum_{j=1}^n (-1)^{1+j} a_{1j} \det A_{1j}\end{aligned}$$

The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column. The expansion across the i th row using the cofactors in (4) is

$$\det A = a_{i1} C_{i1} + a_{i2} C_{i2} + \cdots + a_{in} C_{in}$$

The cofactor expansion down the j th column is

$$\det A = a_{1j} C_{1j} + a_{2j} C_{2j} + \cdots + a_{nj} C_{nj}$$

If A is a triangular matrix, then $\det A$ is the product of the entries on the main diagonal of A .

H3.2: Eigenschappen van determinanten

Row Operations

Let A be a square matrix.

- If a multiple of one row of A is added to another row to produce a matrix B , then $\det B = \det A$.
- If two rows of A are interchanged to produce B , then $\det B = -\det A$.
- If one row of A is multiplied by k to produce B , then $\det B = k \cdot \det A$.

$$\det A = \begin{cases} (-1)^r \cdot \left(\text{product of} \right. \\ \quad \left. \text{pivots in } U \right) & \text{when } A \text{ is invertible} \\ 0 & \text{when } A \text{ is not invertible} \end{cases}$$

A square matrix A is invertible if and only if $\det A \neq 0$.

If A is an $n \times n$ matrix, then $\det A^T = \det A$.

Multiplicative Property

If A and B are $n \times n$ matrices, then $\det AB = (\det A)(\det B)$.

H3.3: Regel van Cramer, volume en lineaire transformaties

Cramer's Rule

Let A be an invertible $n \times n$ matrix. For any $\mathbf{b}$ in $\mathbb{R}^n$, the unique solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ has entries given by

$$x_i = \frac{\det A_i(\mathbf{b})}{\det A}, \quad i = 1, 2, \dots, n \quad (1)$$

Stappenplan:

1. Bereken $\det(A)$ en bepaal of A een inverteerbare matrix is (je kan de stelling van Cramer enkel toepassen als de matrix inverteerbaar is).
2. Bepaal $A_i(\mathbf{b})$ voor alle i in $\{1, \dots, n\}$.
3. Bereken $\det(A_i(\mathbf{b}))$ voor alle i in $\{1, \dots, n\}$.
4. Bepaal $x_i = \frac{\det(A_i(\mathbf{b}))}{\det(A)}$ voor alle i in $\{1, \dots, n\}$.

An Inverse Formula

Let A be an invertible $n \times n$ matrix. Then

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A$$

Stappenplan:

1. Bereken $\det(A)$.
2. Bereken C_{ij} voor alle i en j in $\{1, \dots, n\}$.
3. Bepaal $\operatorname{adj}(A)$.
4. Bepaal A^{-1} .

If A is a 2×2 matrix, the area of the parallelogram determined by the columns of A is $|\det A|$. If A is a 3×3 matrix, the volume of the parallelepiped determined by the columns of A is $|\det A|$.

Let $\mathbf{a}_1$ and $\mathbf{a}_2$ be nonzero vectors. Then for any scalar c , the area of the parallelogram determined by $\mathbf{a}_1$ and $\mathbf{a}_2$ equals the area of the parallelogram determined by $\mathbf{a}_1$ and $\mathbf{a}_2 + c\mathbf{a}_1$.

Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the linear transformation determined by a 2×2 matrix A . If S is a parallelogram in $\mathbb{R}^2$, then

$$\{\text{area of } T(S)\} = |\det A| \cdot \{\text{area of } S\} \quad (5)$$

If T is determined by a 3×3 matrix A , and if S is a parallelepiped in $\mathbb{R}^3$, then

$$\{\text{volume of } T(S)\} = |\det A| \cdot \{\text{volume of } S\} \quad (6)$$

Vectorruimten

4.1 Vectorruimten

A **vector space** is a nonempty set V of objects, called *vectors*, on which are defined two operations, called *addition* and *multiplication by scalars* (real numbers), subject to the ten axioms (or rules) listed below.¹ The axioms must hold for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and for all scalars c and d .

1. The sum of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u} + \mathbf{v}$, is in V .
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
4. There is a **zero** vector $\mathbf{0}$ in V such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
5. For each $\mathbf{u}$ in V , there is a vector $-\mathbf{u}$ in V such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
6. The scalar multiple of $\mathbf{u}$ by c , denoted by $c\mathbf{u}$, is in V .
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$.
10. $1\mathbf{u} = \mathbf{u}$.

For each $\mathbf{u}$ in V and scalar c ,

$$0\mathbf{u} = \mathbf{0} \quad (1)$$

$$c\mathbf{0} = \mathbf{0} \quad (2)$$

$$-\mathbf{u} = (-1)\mathbf{u} \quad (3)$$

Voorbeelden van vectorruimten:

- De kolomruimte $\mathbb{R}^n$

$$\mathbb{R}^n = \left\{ \left[\begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array} \right] \mid x_1, x_2, \dots, x_n \in \mathbb{R} \right\}$$

- $M_{m \times n}(\mathbb{R})$: de verzameling van alle $m \times n$ matrices over $\mathbb{R}$

$$M_{m \times n}(\mathbb{R}) = \left\{ [a^{ij}]_{1 \leq i \leq m, 1 \leq j \leq n} \mid a^{ij} \in \mathbb{R}, 1 \leq i \leq m, 1 \leq j \leq n \right\}$$

- Veeltermen

$$P_n = \{r_0 + r_1X + \cdots + r_nX^n \mid r_i \in \mathbb{R}\}$$

- Differentiaalvergelijkingen

$$S = \{Ae^x + Be^{-x} \mid A, B \in \mathbb{R}\}.$$

- Continue functies op een gesloten interval

Gegeven een gesloten interval $[a, b] \subset \mathbb{R}, a \neq$

b , b.v. $[0, 1]$. Definieer $\mathcal{C}([a, b]) = \{f : [a, b] \rightarrow \mathbb{R} \mid f \text{ is continu over } [a, b]\}$

A **subspace** of a vector space V is a subset H of V that has three properties:

- The zero vector of V is in H .²
- H is closed under vector addition. That is, for each $\mathbf{u}$ and $\mathbf{v}$ in H , the sum $\mathbf{u} + \mathbf{v}$ is in H .
- H is closed under multiplication by scalars. That is, for each $\mathbf{u}$ in H and each scalar c , the vector $c\mathbf{u}$ is in H .

Voorwaarde b en c zijn gelijkwaardig met:

Voor elke twee vectoren $\mathbf{u}$ en $\mathbf{v}$
in H en voor elke twee scalaren c en d in $\mathbb{R}$ geldt $c\mathbf{u} + d\mathbf{v} \in H$.

If $\mathbf{v}_1, \dots, \mathbf{v}_p$ are in a vector space V , then $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is a subspace of V .

EXAMPLE: Is the set H of all matrices of the form

$\begin{bmatrix} 2a & b \\ 3a+b & 3b \end{bmatrix}$ a subspace of $M_{2 \times 2}$? Explain.

Solution: Since

$$\begin{bmatrix} 2a & b \\ 3a+b & 3b \end{bmatrix} = \begin{bmatrix} 2a & 0 \\ 3a & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \\ b & 3b \end{bmatrix}$$

$$= a \begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix}.$$

Therefore $H = \text{Span} \left\{ \begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix} \right\}$ and so H is a subspace of $M_{2 \times 2}$.

4.2 Nulruimten, kolomruimten en lineaire transformaties

The **null space** of an $m \times n$ matrix A , written as $\text{Nul } A$, is the set of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$. In set notation,

$$\text{Nul } A = \{\mathbf{x} : \mathbf{x} \text{ is in } \mathbb{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\}$$

The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions to a system $A\mathbf{x} = \mathbf{0}$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbb{R}^n$.

Bereken $\text{Nul}(A)$ voor $A = \begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 6 & 12 & 13 & 0 & 3 \end{bmatrix}$. Dus wij zoeken $\mathbf{x}$

zodat $A\mathbf{x} = \vec{0}$.

Oplossing:

$$\begin{bmatrix} 3 & 6 & 6 & 3 & 9 & 0 \\ 6 & 12 & 13 & 0 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 6 & 6 & 3 & 9 & 0 \\ 0 & 0 & 1 & -6 & -15 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 2 & 1 & 3 & 0 \\ 0 & 0 & 1 & -6 & -15 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 13 & 33 & 0 \\ 0 & 0 & 1 & -6 & -15 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 - 13x_4 - 33x_5 \\ x_2 \\ 6x_4 + 15x_5 \\ x_4 \\ x_5 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 - 13x_4 - 33x_5 \\ x_2 \\ 6x_4 + 15x_5 \\ x_4 \\ x_5 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -13 \\ 0 \\ 6 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -33 \\ 0 \\ 15 \\ 0 \\ 1 \end{bmatrix}$$

Dus

$$\text{Nul}(A) = \text{Span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\},$$

evenveel generatoren als vrije veranderlijken en deze vectoren zijn (via deze methode) ook **lineair onafhankelijk**.

The **column space** of an $m \times n$ matrix A , written as $\text{Col } A$, is the set of all linear combinations of the columns of A . If $A = [\mathbf{a}_1 \ \cdots \ \mathbf{a}_n]$, then

$$\text{Col } A = \text{Span}\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$$

The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.

The column space of an $m \times n$ matrix A is all of $\mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbb{R}^m$.

A **linear transformation** T from a vector space V into a vector space W is a rule that assigns to each vector $\mathbf{x}$ in V a unique vector $T(\mathbf{x})$ in W , such that

- (i) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in V , and
- (ii) $T(c\mathbf{u}) = cT(\mathbf{u})$ for all $\mathbf{u}$ in V and all scalars c .

Definitie

Het beeld (range) van T is

$$T(V) = \{T(v) \mid v \in V\}$$

Eigenschap

Als T een matrix afbeelding is met geassocieerde matrix A , dan is $T(V) = \text{Col}(A)$.

Definitie

De kern van een lineaire afbeelding $T: V \rightarrow W$ is

$$\text{kern}(T) = \{v \mid v \in V \text{ en } T(v) = 0\}$$

Eigenschap

Als $T: V \rightarrow W$ is een lineaire afbeelding. Dan is $\text{kern}(T)$ een deelruimte van V .

4.3 Lineair onafhankelijke verzamelingen en basis

An indexed set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ of two or more vectors, with $\mathbf{v}_1 \neq \mathbf{0}$, is linearly dependent if and only if some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors, $\mathbf{v}_1, \dots, \mathbf{v}_{j-1}$.

Gevolg

Een verzameling die de nulvector $\mathbf{0}$ bevat is lineair afhankelijk.

Gevolg

Een verzameling met twee vectoren is lineair afhankelijk als een van de twee vectoren een veelvoud is van de andere vector.

Let H be a subspace of a vector space V . An indexed set of vectors $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_p\}$ in V is a **basis** for H if

- (i) $\mathcal{B}$ is a linearly independent set, and
- (ii) the subspace spanned by $\mathcal{B}$ coincides with H ; that is,

$$H = \text{Span}\{\mathbf{b}_1, \dots, \mathbf{b}_p\}$$

Gegeven vectoren en je moet checken of het een basis is voor $\mathbb{R}^n$:

1. Voeg de vectoren samen in 1 matrix met een nulkolom
2. Werk uit naar de gereduceerde echelonvorm
3. De kolommen met de pivot posities zijn dan een basis

Gevolg

Een basis voor $\mathbb{R}^n$ bevat steeds precies n vectoren, want:

- ▶ een verzameling met minstens $n + 1$ vectoren is steeds een lineair afhankelijk stel vectoren in $\mathbb{R}^n$
- ▶ een verzameling met hoogstens $n - 1$ vectoren spant de ruimte $\mathbb{R}^n$ zeker niet op.

The Spanning Set Theorem

Let $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ be a set in V , and let $H = \text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$.

- a. If one of the vectors in S —say, $\mathbf{v}_k$ —is a linear combination of the remaining vectors in S , then the set formed from S by removing $\mathbf{v}_k$ still spans H .
- b. If $H \neq \{\mathbf{0}\}$, some subset of S is a basis for H .

The pivot columns of a matrix A form a basis for $\text{Col } A$.

Gegeven

$$A = [\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3 \ \mathbf{a}_4] = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix}$$

Gevraagd: een basis voor de nul- en kolomruimte van de matrix A .

Oplossing

$$A \sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & -1 & -5 \\ 0 & 0 & 2 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

pivotkolom 1, 3.

Oplossing

$$A \sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & -1 & -5 \\ 0 & 0 & 2 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Dus de oplossingenverzameling van $A\mathbf{x} = \mathbf{0}$ is

$$S = \left\{ \begin{bmatrix} -2x_2 - 4x_4 \\ x_2 \\ -5x_4 \\ x_4 \end{bmatrix} \right\} = \text{Span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ -5 \\ 1 \end{bmatrix} \right\} = \text{Nul}(A),$$

en

$$\text{Col}(A) = \text{Span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \\ 0 \end{bmatrix} \right\} \quad \textit{originele matrix}$$

4.4 Coördinaatsystemen

The Unique Representation Theorem

Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be a basis for a vector space V . Then for each $\mathbf{x}$ in V , there exists a unique set of scalars $c_1, \dots, c_n$ such that

$$\mathbf{x} = c_1 \mathbf{b}_1 + \dots + c_n \mathbf{b}_n \quad (1)$$

Suppose $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ is a basis for V and $\mathbf{x}$ is in V . The **coordinates of $\mathbf{x}$ relative to the basis $\mathcal{B}$** (or the **$\mathcal{B}$ -coordinates of $\mathbf{x}$**) are the weights $c_1, \dots, c_n$ such that $\mathbf{x} = c_1 \mathbf{b}_1 + \dots + c_n \mathbf{b}_n$.

Gegeven $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ met $\mathbf{b}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Gevraagd: bepaal de vector $\mathbf{x}$ waarvoor $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

Oplossing

$$\mathbf{x} = 2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$$

Wat zijn de coördinaten van $\mathbf{x}$ ten opzichte van de standaardbasis $\mathcal{E}$ van $\mathbb{R}^2$?

Oplossing

$$[\mathbf{x}]_{\mathcal{E}} = \begin{bmatrix} 6 \\ 5 \end{bmatrix} . \quad \vec{x} = 6\vec{e}_1 + 5\vec{e}_2$$

Gegeven $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ met $\mathbf{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ en $\mathbf{x} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$.

Gevraagd: bepaal de coördinaatvector $[\mathbf{x}]_{\mathcal{B}}$

Oplossing

Als $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, dan moet

$$\vec{x} \stackrel{!}{=} 3 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \stackrel{!}{=} \vec{x}$$

$$c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix},$$

of nog,

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}.$$

Dus

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$P_{\mathcal{B}} = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_n]$$

Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be a basis for a vector space V . Then the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is a one-to-one linear transformation from V onto $\mathbb{R}^n$.

EXAMPLE Let $\beta = \langle \mathbf{b}_1, \mathbf{b}_2 \rangle$ where $\mathbf{b}_1 = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$ and

$\mathbf{b}_2 = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$ and let $H = \text{span}\langle \mathbf{b}_1, \mathbf{b}_2 \rangle$. Find $[\mathbf{x}]_{\beta}$, if $\mathbf{x} = \begin{bmatrix} 9 \\ 13 \\ 15 \end{bmatrix}$.

Solution: (a) Find c_1 and c_2 such that

$$c_1 \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 13 \\ 15 \end{bmatrix}$$

Corresponding augmented matrix:

$$\left[\begin{array}{ccc|c} 3 & 0 & 9 \\ 3 & 1 & 13 \\ 1 & 3 & 15 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{array} \right]$$

Dus $c_1 = 3$ en $c_2 = 4$

$$\text{en } [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Voorbeeld: matrices als vectoren

Beschouw de vectorruimte $V = M_{2 \times 2}(\mathbb{R})$. Stel $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4\}$ met

$$\mathbf{b}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \mathbf{b}_3 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \mathbf{b}_4 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

1. Controleer dat $\mathcal{B}$ een basis is van V
2. Bepaal $[\mathbf{a}]_{\mathcal{B}}$ met $\mathbf{a} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.
3. Bepaal de coördinaat afbeelding $T : V \rightarrow \mathbb{R}^4$ ten opzichte van de basis $\mathcal{B}$.

1. Veronderstel dat $x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + x_3\mathbf{b}_3 + x_4\mathbf{b}_4 = \mathbf{0}$. Dit is gelijkwaardig met

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_2 + x_3 + x_4 = 0 \\ x_3 + x_4 = 0 \\ x_4 = 0 \end{cases} \iff \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Handwritten notes: Above the matrix A, there are blue annotations: $x_1 + x_2 + x_3 + x_4$ above the first row, $x_2 + x_3 + x_4$ above the second row, $x_3 + x_4$ above the third row, and x_4 above the fourth row. To the right of the matrix equation, there is a blue bracketed vector $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and another blue bracketed vector $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ with an arrow pointing to it from the first vector.

Nu is duidelijk dat A 4 spilelementen heeft. Dat betekent dat bovenstaande matrixvergelijking enkel de nuloplossing heeft, dus $\mathcal{B}$ is een stel lineair onafhankelijke vectoren. En er volgt ook dat er voor elke matrix $B \in V$ een uniek viertal (x_1, x_2, x_3, x_4) bestaat zodat $B = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + x_3\mathbf{b}_3 + x_4\mathbf{b}_4$, of dus $\text{Span}(\mathcal{B}) = V$.

2. De matrixvergelijking die opgelost moet worden is

$$x_1\vec{b}_1 + x_2\vec{b}_2 + x_3\vec{b}_3 + x_4\vec{b}_4 = \vec{a} \quad \text{with} \quad \vec{a} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

Handwritten notes: Above the equation, there are blue annotations: $x_1\vec{b}_1 + x_2\vec{b}_2 + x_3\vec{b}_3 + x_4\vec{b}_4$ above the left side, and $\vec{a} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ above the right side. A blue arrow points from the left side of the equation to the matrix A in the previous block.

Deze herleidt zich onmiddellijk tot

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow [\mathbf{a}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ -1 \\ -1 \\ 4 \end{bmatrix}$$

Inderdaad

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + 4 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

3. Beschouw $\mathbf{v} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in V$. Dan is

$$\begin{bmatrix} 1 & 1 & 1 & 1 & a \\ 0 & 1 & 1 & 1 & b \\ 0 & 0 & 1 & 1 & c \\ 0 & 0 & 0 & 1 & d \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & a-b \\ 0 & 1 & 0 & 0 & b-c \\ 0 & 0 & 1 & 0 & c-d \\ 0 & 0 & 0 & 1 & d \end{bmatrix} \Rightarrow [\mathbf{a}]_{\mathcal{B}} = \begin{bmatrix} a-b \\ b-c \\ c-d \\ d \end{bmatrix}$$

Dus

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a-b \\ b-c \\ c-d \\ d \end{bmatrix}.$$

4.5 Dimensie van een vectorruimte

If a vector space V has a basis $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$, then any set in V containing more than n vectors must be linearly dependent.

If a vector space V has a basis of n vectors, then every basis of V must consist of exactly n vectors.

If V is spanned by a finite set, then V is said to be **finite-dimensional**, and the **dimension** of V , written as $\dim V$, is the number of vectors in a basis for V . The dimension of the zero vector space $\{\mathbf{0}\}$ is defined to be zero. If V is not spanned by a finite set, then V is said to be **infinite-dimensional**.

de vectorruimte $\mathbb{R}^2$, met

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}.$$

alternatieve basis (oefening)

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

De dimensie van $\mathbb{R}^2$ is 2.

dimensie van $\mathbb{R}^n$ is $n \in \mathbb{N}$

De vectorruimte $P_2 = \{a_0 + a_1X + a_2X^2 \mid a_0, a_1, a_2 \in \mathbb{R}\}$ met

$$\mathcal{B} = \{1, X, X^2\}.$$

De dimensie van P_2 is 3. Algemeen, de dimensie van P_n is $n + 1$.

De vectorruimte $P = \{a_0 + a_1X + \dots + a_dX^d \mid a_0, a_1, \dots, a_d \in \mathbb{R}, d \in \mathbb{N}\}$ met

$$\mathcal{B} = \{1, X, X^2, X^3, \dots\}.$$

De vectorruimte P is een oneindigdimensionale vectorruimte.

De vectorruimte $M_{2 \times 2}(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{R} \right\}$ met

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

De dimensie van $M_{2 \times 2}(\mathbb{R})$ is 4. Algemeen, de dimensie van $M_{m \times n}(\mathbb{R})$ is mn .

Let H be a subspace of a finite-dimensional vector space V . Any linearly independent set in H can be expanded, if necessary, to a basis for H . Also, H is finite-dimensional and

$$\dim H \leq \dim V$$

EXAMPLE: Find a basis and the dimension of the subspace

$$W = \left\{ \begin{bmatrix} a + b + 2c \\ 2a + 2b + 4c + d \\ b + c + d \\ 3a + 3c + d \end{bmatrix} : a, b, c, d \text{ are real} \right\}.$$

Solution: Since

$$\begin{bmatrix} a + b + 2c \\ 2a + 2b + 4c + d \\ b + c + d \\ 3a + 3c + d \end{bmatrix} = a \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix} + b \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 2 \\ 4 \\ 1 \\ 3 \end{bmatrix} + d \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

$$W = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$$

Oefening

Ga na dat de vectoren $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ een lineair afhankelijk stel vormen en bepaal een basis voor $W = \text{Col}(A)$ met A de matrix die deze vectoren als kolommen heeft.

Oplossing

$$A = \begin{bmatrix} 1 & 1 & 2 & 0 \\ 2 & 2 & 4 & 1 \\ 0 & 1 & 1 & 1 \\ 3 & 0 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & -3 & -3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix} \sim$$
$$\begin{bmatrix} \textcircled{1} & 1 & 2 & 0 \\ 0 & \textcircled{1} & 1 & 1 \\ 0 & 0 & 0 & \textcircled{1} \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad 3 \text{ spilelemente.}$$

Dus

$$W = \text{Span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

$$\Rightarrow \dim W = 3$$

The Basis Theorem

Let V be a p -dimensional vector space, $p \geq 1$. Any linearly independent set of exactly p elements in V is automatically a basis for V . Any set of exactly p elements that spans V is automatically a basis for V .

The dimension of $\text{Nul } A$ is the number of free variables in the equation $A\mathbf{x} = \mathbf{0}$, and the dimension of $\text{Col } A$ is the number of pivot columns in A .

Dimensions of Col A and Nul A

Recall our techniques to find basis sets for column spaces and null spaces.

EXAMPLE: Suppose $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 8 \end{bmatrix}$. Find $\dim \text{Col } A$ and

$\dim \text{Nul } A$.

Solution

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

So $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \end{bmatrix} \right\}$ is a basis for $\text{Col } A$ and $\dim \text{Col } A = 2$.

*erste zwei Spalten
→ 2 = dim Nul(A)*

2 Spalten.

4.6 Rang van een matrix

If two matrices A and B are row equivalent, then their row spaces are the same. If B is in echelon form, the nonzero rows of B form a basis for the row space of A as well as for that of B .

Stelling

- De rijruimte $\text{Row}(A)$ is isomorf met $\text{Col}(A^T)$.
- Als de matrices A en B rij-equivalent zijn, dan geldt
$$\text{Row}(A) = \text{Row}(B)$$
- Als de matrix B in echelonvorm is, dan vormen de niet nul rijen van B een basis van $\text{Row}(A)$.

Voorbeeld

Bepaal de rijruimte van $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & -3 & -2 \end{bmatrix}$.
 $\rightarrow \text{Row}(A) \cong \text{Col}(A^T)$

Oplossing 1: via de kolomruimte van A^T

$$A^T = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & -3 \\ 3 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -5 \\ 0 & 1 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{bmatrix}$$

Dus

$$\text{Col}(A^T) = \text{Span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

of nog

$$\text{Row}(A) = \text{Span} \{ [1 \ 2 \ 3], [0 \ 1 \ 1] \}$$

Oplossing 2: rij-reductie van A

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & -3 & -2 \end{bmatrix} \sim A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -5 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Dus

$$\text{Row}(A) = \text{Span} \{ [1 \ 2 \ 3], [0 \ 1 \ 1] \}$$

The **rank** of A is the dimension of the column space of A .

Stelling

- $\text{rank}(A) = \dim \text{Col}(A) = \dim \text{Row}(A) =$
aantal spil (pivot) kolommen in A .
- $\dim \text{Nul}(A) =$ aantal niet spil kolommen van $A =$
aantal vrije veranderlijken.

$$\text{rank}(A) + \dim \text{Nul}(A) = n.$$

The Rank Theorem

The dimensions of the column space and the row space of an $m \times n$ matrix A are equal. This common dimension, the rank of A , also equals the number of pivot positions in A and satisfies the equation

$$\text{rank } A + \dim \text{Nul } A = n$$

Voorbeeld

Bepaal een basis voor de rijruimte, een basis voor de kolomruimte, de dimensie van de nulruimte en de rang van de matrix A , en controleer voorgaande stelling.

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

Oplossing

$$\begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ \textcircled{1} & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 3 & -5 & 1 & 5 \\ -2 & -5 & 8 & 0 & -17 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 3 & -5 & 1 & 5 \\ 0 & \textcircled{1} & -2 & 2 & -7 \\ 0 & 2 & -4 & 4 & -14 \\ 0 & 4 & -8 & 4 & -8 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 3 & -5 & 1 & 5 \\ 0 & \textcircled{1} & -2 & 2 & -7 \\ 0 & 0 & 0 & \textcircled{1} & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \left\{ \begin{array}{l} \text{basis} \\ \text{Row}(A) \end{array} \right.$$

Dus dan is de basis B , respectievelijk C voor de kolomruimte, respectievelijk de rijruimte,

$$B = \left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 11 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 7 \\ 5 \end{bmatrix} \right\}, \quad C = \left\{ \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}, \right\}$$

Dus:

Probleem Een wetenschapper lost een homogeen lineair systeem op van 50 vergelijkingen in 54 onbekenden en ontdekt dat er 4 vrije veranderlijken zijn. Kan de wetenschapper er zeker van zijn dat elk geassocieerd niet-homogeen lineair systeem (met dezelfde coëfficiënten) een oplossing heeft? **Oplossing** Zij A de coëfficiëntenmatrix. Dan

$$\text{rank}(A) = \dim \text{col}(A) = \# \text{spilkolommen}$$

$$\dim \text{Nul}(A) = \# \text{aantal vrije veranderlijken} = 4$$

Dus

$$\text{rank}(A) + 4 = 54$$

of

$$\text{rank}(A) = 50 = \# \text{vergelijkingen}$$

Dus elk systeem $Ax = b$ heeft een oplossing.

$$L_0 \in \mathbb{R}^{50}$$

The Invertible Matrix Theorem (continued)

Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.

- m. The columns of A form a basis of $\mathbb{R}^n$.
- n. $\text{Col } A = \mathbb{R}^n$
- o. $\dim \text{Col } A = n$
- p. $\text{rank } A = n$
- q. $\text{Nul } A = \{0\}$
- r. $\dim \text{Nul } A = 0$

4.7 Verandering van basis

Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ be bases of a vector space V . Then there is a unique $n \times n$ matrix ${}_{\mathcal{C} \leftarrow \mathcal{B}}^P$ such that

$$[\mathbf{x}]_{\mathcal{C}} = {}_{\mathcal{C} \leftarrow \mathcal{B}}^P [\mathbf{x}]_{\mathcal{B}} \quad (4)$$

The columns of ${}_{\mathcal{C} \leftarrow \mathcal{B}}^P$ are the $\mathcal{C}$ -coordinate vectors of the vectors in the basis $\mathcal{B}$. That is,

$${}_{\mathcal{C} \leftarrow \mathcal{B}}^P = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}} & \cdots & [\mathbf{b}_n]_{\mathcal{C}} \end{bmatrix} \quad (5)$$

Vraag

Gegeven de vectoren

$$\mathbf{b}_1 = \begin{bmatrix} -9 \\ 1 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}, \mathbf{c}_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}, \mathbf{c}_2 = \begin{bmatrix} 3 \\ 5 \end{bmatrix},$$

en

$$\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}, \mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$$

Bepaal de matrix ${}_{\mathcal{C} \leftarrow \mathcal{B}}^P$.

Oplossing

De kolommen van de gezochte matrix zijn de coördinaatvectoren van $\mathbf{b}_1$ en $\mathbf{b}_2$ ten opzichte van de basis $\mathcal{C}$. Er geldt

$$\mathbf{b}_1 = \begin{bmatrix} 1 & 3 \\ -4 & -5 \end{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} = \begin{bmatrix} -9 \\ 1 \end{bmatrix}$$

$$\mathbf{b}_2 = \begin{bmatrix} 1 & 3 \\ -4 & -5 \end{bmatrix} [\mathbf{b}_2]_{\mathcal{C}} = \begin{bmatrix} -5 \\ 1 \end{bmatrix}$$

Dus we moeten twee matrixvergelijkingen oplossen. ...

$$\begin{bmatrix} 1 & 3 & -9 & -5 \\ -4 & -5 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -9 & -5 \\ 0 & 7 & -35 & -21 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -9 & -5 \\ 0 & 1 & -5 & -3 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 0 & 6 & 4 \\ 0 & 1 & -5 & -3 \end{bmatrix}$$

Dus

$$[\mathbf{b}_1]_C = \begin{bmatrix} 6 \\ -5 \end{bmatrix}, [\mathbf{b}_2]_C = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$$

en

$$P_{C \leftarrow B} = \begin{bmatrix} 6 & 4 \\ -5 & -3 \end{bmatrix}$$

Vraag

Stel $\vec{x} = \vec{b}_1 + \vec{b}_2$. Wat is dan $[\vec{x}]_C$?

Oplossing

Stel

$$\mathbf{x} = \mathbf{b}_1 + \mathbf{b}_2 = \begin{bmatrix} -14 \\ 0 \end{bmatrix}, \text{ dus } [\mathbf{x}]_B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Nu is

$$[\mathbf{x}]_C = P_{C \leftarrow B} [\mathbf{x}]_B = \begin{bmatrix} 6 & 4 \\ -5 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ -8 \end{bmatrix}$$

en inderdaad

$$10\mathbf{c}_1 - 8\mathbf{c}_2 = 10 \begin{bmatrix} 1 \\ -4 \end{bmatrix} - 8 \begin{bmatrix} 3 \\ -5 \end{bmatrix} = \mathbf{x} = \begin{bmatrix} -14 \\ 0 \end{bmatrix} \quad \checkmark$$

$$(P_{C \leftarrow B})^{-1} = P_{B \leftarrow C}$$

4.9 Toepassingen: Markov ketens

If P is an $n \times n$ regular stochastic matrix, then P has a unique steady-state vector $\mathbf{q}$. Further, if $\mathbf{x}_0$ is any initial state and $\mathbf{x}_{k+1} = P\mathbf{x}_k$ for $k = 0, 1, 2, \dots$, then the Markov chain $\{\mathbf{x}_k\}$ converges to $\mathbf{q}$ as $k \rightarrow \infty$.

Met behulp van een Markov keten kan een reeks "metingen" gemodelleerd worden. Er is een eindige verzameling "toestanden" en het resultaat van elke meting is enkel afhankelijk van de vorige meting. De toestanden evolueren volgens vooraf vastgelegde parameters. Het resultaat van de meting van alle toestanden is een vector, de evolutie wordt bepaald door deze te vermenigvuldigen met een matrix.

Voorbeeld

In een stedelijke metropool wordt elk jaar het percentage inwoners in de stad en in alle voorsteden samen gemeten. Het resultaat bestaat uit twee reële getallen. Deze vormen de toestandsvector. Elk jaar blijft 95 % van de bevolking die in de stad woont, daar ook wonen, en 5 % van de bevolking in de stad verhuist naar de voorsteden. Vanuit de voorsteden trekt elk jaar 3 % naar de stad en blijft 97 % in de voorsteden wonen. Het verband tussen toestandsvector $\mathbf{x}_{i+1}$ en $\mathbf{x}_i$ is dan ook

$$\mathbf{x}_{i+1} = \begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix} \mathbf{x}_i$$

Wanneer we op tijdstip 0 een verdeling van 60/40 % stad/voorsteden hebben, dan geldt

$$\mathbf{x}_1 = \begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix} \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} = \begin{bmatrix} 0.565 \\ 0.435 \end{bmatrix}$$

De vraag stelt zich of we aan de hand van een dergelijke Markov keten de toestand op lange termijn kunnen voorspellen.

$$\vec{x}_k = P \vec{x}_{k-1} = P P \vec{x}_{k-2} = \dots = P^k \vec{x}_0$$

$\lim_{k \rightarrow \infty} P^k = ?$

grote $k \in \mathbb{N}$
grote matrix.

Definitie

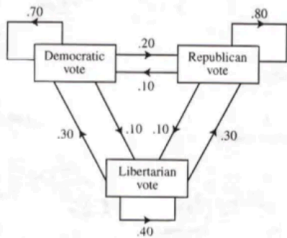
Een vector is een stochastische vector als alle componenten niet-negatief zijn en de som van alle componenten gelijk is aan 1. Een stochastische matrix is een matrix waarvan alle kolommen een stochastische vector zijn.

Eigenschap

Stel M is een stochastische $n \times n$ matrix en $x \in \mathbb{R}^n$ een stochastische vector. Dan is Mx opnieuw een stochastische vector.

Gevolg

Het product van twee stochastische $n \times n$ matrices is opnieuw een stochastische $n \times n$ matrix.



Figuur: Verloop van stemmen

$$P = \begin{matrix} & \begin{matrix} D & R & L \end{matrix} \\ \begin{matrix} D \\ R \\ L \end{matrix} & \begin{bmatrix} 0.7 & 0.1 & 0.3 \\ 0.2 & 0.8 & 0.3 \\ 0.1 & 0.1 & 0.4 \end{bmatrix} \end{matrix}$$

*P_{ij}: percentages
van kiesers
van i → j*

Verloop van volgende verkiezingen als

$$\mathbf{x}_0 = \begin{bmatrix} 0.55 \\ 0.40 \\ 0.05 \end{bmatrix}$$

$$\mathbf{x}_1 = P\mathbf{x}_0 = \begin{bmatrix} .70 & .10 & .30 \\ .20 & .80 & .30 \\ .10 & .10 & .40 \end{bmatrix} \begin{bmatrix} .55 \\ .40 \\ .05 \end{bmatrix} = \begin{bmatrix} .440 \\ .445 \\ .115 \end{bmatrix}$$

$$\mathbf{x}_2 = P\mathbf{x}_1 = \begin{bmatrix} .70 & .10 & .30 \\ .20 & .80 & .30 \\ .10 & .10 & .40 \end{bmatrix} \begin{bmatrix} .440 \\ .445 \\ .115 \end{bmatrix} = \begin{bmatrix} .3870 \\ .4785 \\ .1345 \end{bmatrix}$$

EXAMPLE 3 Let $P = \begin{bmatrix} .5 & .2 & .3 \\ .3 & .8 & .3 \\ .2 & 0 & .4 \end{bmatrix}$ and $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

SOLUTION

$$\mathbf{x}_1 = P\mathbf{x}_0 = \begin{bmatrix} .5 & .2 & .3 \\ .3 & .8 & .3 \\ .2 & 0 & .4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} .5 \\ .3 \\ .2 \end{bmatrix}$$

$$\mathbf{x}_2 = P\mathbf{x}_1 = \begin{bmatrix} .5 & .2 & .3 \\ .3 & .8 & .3 \\ .2 & 0 & .4 \end{bmatrix} \begin{bmatrix} .5 \\ .3 \\ .2 \end{bmatrix} = \begin{bmatrix} .37 \\ .45 \\ .18 \end{bmatrix}$$

$$\mathbf{x}_3 = P\mathbf{x}_2 = \begin{bmatrix} .5 & .2 & .3 \\ .3 & .8 & .3 \\ .2 & 0 & .4 \end{bmatrix} \begin{bmatrix} .37 \\ .45 \\ .18 \end{bmatrix} = \begin{bmatrix} .329 \\ .525 \\ .146 \end{bmatrix}$$

The results of further calculations are shown below, with entries rounded to four or five significant figures.

$$\mathbf{x}_4 = \begin{bmatrix} .3133 \\ .5625 \\ .1242 \end{bmatrix}, \quad \mathbf{x}_5 = \begin{bmatrix} .3064 \\ .5813 \\ .1123 \end{bmatrix}, \quad \mathbf{x}_6 = \begin{bmatrix} .3032 \\ .5906 \\ .1062 \end{bmatrix}, \quad \mathbf{x}_7 = \begin{bmatrix} .3016 \\ .5953 \\ .1031 \end{bmatrix}$$

$$\mathbf{x}_8 = \begin{bmatrix} .3008 \\ .5977 \\ .1016 \end{bmatrix}, \quad \mathbf{x}_9 = \begin{bmatrix} .3004 \\ .5988 \\ .1008 \end{bmatrix}, \quad \mathbf{x}_{10} = \begin{bmatrix} .3002 \\ .5994 \\ .1004 \end{bmatrix}, \quad \mathbf{x}_{11} = \begin{bmatrix} .3001 \\ .5997 \\ .1002 \end{bmatrix}$$

$$\mathbf{x}_{12} = \begin{bmatrix} .30005 \\ .59985 \\ .10010 \end{bmatrix}, \quad \mathbf{x}_{13} = \begin{bmatrix} .30002 \\ .59993 \\ .10005 \end{bmatrix}, \quad \mathbf{x}_{14} = \begin{bmatrix} .30001 \\ .59996 \\ .10002 \end{bmatrix}, \quad \mathbf{x}_{15} = \begin{bmatrix} .30001 \\ .59998 \\ .10001 \end{bmatrix}$$

$$P\mathbf{q} = \begin{bmatrix} .5 & .2 & .3 \\ .3 & .8 & .3 \\ .2 & 0 & .4 \end{bmatrix} \begin{bmatrix} .3 \\ .6 \\ .1 \end{bmatrix} = \begin{bmatrix} .15 + .12 + .03 \\ .09 + .48 + .03 \\ .06 + 0 + .04 \end{bmatrix} = \begin{bmatrix} .30 \\ .60 \\ .10 \end{bmatrix} = \mathbf{q}$$

Definitie

Veronderstel dat P een $n \times n$ stochastische matrix is. Een vector q waarvoor geldt dat

$$Pq = q$$

is een stationaire vector door P .

EXAMPLE 4 The probability vector $q = \begin{bmatrix} .375 \\ .625 \end{bmatrix}$ is a steady-state vector for the population migration matrix M in Example 1, because

$$Mq = \begin{bmatrix} .95 & .03 \\ .05 & .97 \end{bmatrix} \begin{bmatrix} .375 \\ .625 \end{bmatrix} = \begin{bmatrix} .35625 + .01875 \\ .01875 + .60625 \end{bmatrix} = \begin{bmatrix} .375 \\ .625 \end{bmatrix} = q \quad \blacksquare$$

EXAMPLE 5 Let $P = \begin{bmatrix} .6 & .3 \\ .4 & .7 \end{bmatrix}$. Find a steady-state vector for P .

SOLUTION First, solve the equation $Px = x$.

$$\begin{aligned} Px - x &= 0 \\ Px - Ix &= 0 && \text{Recall from Section 1.4 that } Ix = x. \\ (P - I)x &= 0 \end{aligned}$$

For P as above,

$$P - I = \begin{bmatrix} .6 & .3 \\ .4 & .7 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -.4 & .3 \\ .4 & -.3 \end{bmatrix}$$

To find all solutions of $(P - I)x = 0$, row reduce the augmented matrix:

$$\left[\begin{array}{ccc|c} -.4 & .3 & 0 & 0 \\ .4 & -.3 & 0 & 0 \end{array} \right] \sim \left[\begin{array}{ccc|c} -.4 & .3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & -3/4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Then $x_1 = \frac{3}{4}x_2$ and x_2 is free. The general solution is $x_2 \begin{bmatrix} 3/4 \\ 1 \end{bmatrix}$.

Next, choose a simple basis for the solution space. One obvious choice is $\begin{bmatrix} 3/4 \\ 1 \end{bmatrix}$

but a better choice with no fractions is $\mathbf{w} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ (corresponding to $x_2 = 4$).

Finally, find a probability vector in the set of all solutions of $P\mathbf{x} = \mathbf{x}$. This process is easy, since every solution is a multiple of the solution $\mathbf{w}$ above. Divide $\mathbf{w}$ by the sum of its entries and obtain

$$\mathbf{q} = \begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix}$$

As a check, compute

$$P\mathbf{q} = \begin{bmatrix} 6/10 & 3/10 \\ 4/10 & 7/10 \end{bmatrix} \begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix} = \begin{bmatrix} 18/70 + 12/70 \\ 12/70 + 28/70 \end{bmatrix} = \begin{bmatrix} 30/70 \\ 40/70 \end{bmatrix} = \mathbf{q} \quad \blacksquare$$

Stelling

Voor een stochastische $n \times n$ matrix bestaat er altijd een stationaire vector.

Eigenwaarden en eigenvectoren

An **eigenvector** of an $n \times n$ matrix A is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ . A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution $\mathbf{x}$ of $A\mathbf{x} = \lambda\mathbf{x}$; such an $\mathbf{x}$ is called an *eigenvector corresponding to λ* .¹

Om λ te vinden moet $(A - \lambda I_n)\vec{x} = \vec{0}$ een niet-triviale oplossing hebben. Als $(A - \lambda I_n)$ inverteerbaar is, dan is $\vec{x} = (A - \lambda I_n)^{-1}\vec{0} = \vec{0}$. Dus $(A - \lambda I_n)$ mag niet inverteerbaar zijn. Uit H2.2 Theorem 4 weten we dat $(A - \lambda I_n)$ niet inverteerbaar is als en slechts als $\det(A - \lambda I_n) = 0$. We zoeken dus λ zodat $\det(A - \lambda I_n) = 0$.

Als we λ vinden, zijn de bijhorende eigenvectoren $\{\vec{x} \in \mathbb{R}^n \mid (A - \lambda I_n)\vec{x} = \vec{0}\} = \text{Nul}(A - \lambda I_n)$. We moeten dus de oplossingsverzameling vinden van $\begin{bmatrix} (A - \lambda I_n) & \vec{0} \end{bmatrix}$.

Thus λ is an eigenvalue of an $n \times n$ matrix A if and only if the equation

$$(A - \lambda I)\mathbf{x} = \mathbf{0} \quad (3)$$

has a nontrivial solution. The set of *all* solutions of (3) is just the null space of the matrix $A - \lambda I$. So this set is a *subspace* of $\mathbb{R}^n$ and is called the **eigenspace** of A corresponding to λ . The eigenspace consists of the zero vector and all the eigenvectors corresponding to λ .

$V_\lambda = \{\vec{x} \in \mathbb{R}^n \mid (A - \lambda I_n)\vec{x} = \vec{0}\}$ is de **eigenruimte** van A horende bij de eigenwaarde λ . Het is de verzameling van de nulvector en alle eigenvectoren horende bij λ .

The eigenvalues of a triangular matrix are the entries on its main diagonal.

If $\mathbf{v}_1, \dots, \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \dots, \lambda_r$ of an $n \times n$ matrix A , then the set $\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$ is linearly independent.

Om λ te vinden moet $(A - \lambda I_n)\vec{x} = \vec{0}$ een niet-triviale oplossing hebben. Als $(A - \lambda I_n)$ inverteerbaar is, dan is $\vec{x} = (A - \lambda I_n)^{-1}\vec{0} = \vec{0}$. Dus $(A - \lambda I_n)$ mag niet inverteerbaar zijn. Uit H2.2 Theorem 4 weten we dat $(A - \lambda I_n)$ niet inverteerbaar is als en slechts als $\det(A - \lambda I_n) = 0$. We zoeken dus λ zodat $\det(A - \lambda I_n) = 0$.

The Invertible Matrix Theorem (continued)

Let A be an $n \times n$ matrix. Then A is invertible if and only if:

- s. The number 0 is *not* an eigenvalue of A .
- t. The determinant of A is *not* zero.

Properties of Determinants

Let A and B be $n \times n$ matrices.

- A is invertible if and only if $\det A \neq 0$.
- $\det AB = (\det A)(\det B)$.
- $\det A^T = \det A$.
- If A is triangular, then $\det A$ is the product of the entries on the main diagonal of A .
- A row replacement operation on A does not change the determinant. A row interchange changes the sign of the determinant. A row scaling also scales the determinant by the same scalar factor.

If $n \times n$ matrices A and B are similar, then they have the same characteristic polynomial and hence the same eigenvalues (with the same multiplicities).

A scalar λ is an eigenvalue of an $n \times n$ matrix A if and only if λ satisfies the characteristic equation

$$\det(A - \lambda I) = 0$$

WARNINGS:

1. The matrices

$$\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

are not similar even though they have the same eigenvalues.

2. Similarity is not the same as row equivalence. (If A is row equivalent to B , then $B = EA$ for some invertible matrix E .) Row operations on a matrix usually change its eigenvalues.

The Diagonalization Theorem

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

In fact, $A = PDP^{-1}$, with D a diagonal matrix, if and only if the columns of P are n linearly independent eigenvectors of A . In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P .

An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \dots, \lambda_p$.

- For $1 \leq k \leq p$, the dimension of the eigenspace for λ_k is less than or equal to the multiplicity of the eigenvalue λ_k .
- The matrix A is diagonalizable if and only if the sum of the dimensions of the eigenspaces equals n , and this happens if and only if (i) the characteristic polynomial factors completely into linear factors and (ii) the dimension of the eigenspace for each λ_k equals the multiplicity of λ_k .
- If A is diagonalizable and $\mathcal{B}_k$ is a basis for the eigenspace corresponding to λ_k for each k , then the total collection of vectors in the sets $\mathcal{B}_1, \dots, \mathcal{B}_p$ forms an eigenvector basis for $\mathbb{R}^n$.

Stappenplan voor het diagonaliseren van een $n \times n$ -matrix A :

- Vind de eigenwaarden van A via de karakteristieke vergelijking $0 = \det(A - \lambda I_n)$.
- Vind voor elke eigenwaarde de bijhorende verzameling lineair onafhankelijke eigenvectoren door de vergelijking $(A - \lambda I_n)\mathbf{x} = 0$ op te lossen. Je moet n lineair onafhankelijke eigenvectoren vinden opdat de matrix A diagonaliseerbaar is.
- Bepaal $P = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \dots \quad \mathbf{v}_n]$ met de gevonden eigenvectoren uit stap 2.
- Bepaal $D = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$ met de gevonden eigenwaarden uit stap 2, in dezelfde volgorde als de eigenvectoren in P . We hebben dat $A = PDP^{-1}$.

Theorie Vragen

Lineaire Algebra

Vragen (V of W)

- 1 V Een vierkante matrix die diagonaliseerbaar is, is ook inverseerbaar.
 - 2 V Elke vierkante matrix is diagonaliseerbaar.
 - 3 V Elke vierkante matrix is inverseerbaar.
 - 4 W Stel dat A een 3×3 matrix is. Dan is A inverseerbaar als de lineaire transformatie horende bij A injectief en surjectief is.
 - 5 V Stel dat A , B en C $n \times n$ matrices zijn. Als $AB = AC$ dan volgt $B = C$.
 - 6 V Zij V een vectorruimte. Als er een lineair afhankelijke verzameling $\{\vec{v}_1, \dots, \vec{v}_p\}$ bestaat in V , dan is $\dim V \leq p$.
 - 7 V Stel A is een 3×4 matrix. Dan is de dimensie van de kolomruimte van A gelijk aan 4.
 - 8 W Stel A is een 3×4 matrix. Dan is de dimensie van de rijruimte van A hoogstens 3.
 - 9 W Als het systeem $A\vec{x} = \vec{b}$ meer dan 1 oplossing heeft, dan geldt dit ook voor het systeem $A\vec{x} = \vec{0}$.
 - 10 V De rang van een 3×5 matrix kan gelijk zijn aan 5.
-
- 1 W Als het systeem $A\vec{x} = \vec{b}$ oneindig veel oplossingen heeft, dan heeft het systeem $A\vec{x} = \vec{0}$ ook oneindig veel oplossingen.
 - 2 W Een 3×3 matrix waarvan de kolomruimte 2-dimensionaal is, heeft determinant 0.
 - 3 W Stel A is een 3×3 matrix met twee verschillende eigenwaarden, namelijk -1 en 1 . Dan is zijn determinant -1 of 1 .
 - 4 W Een basis voor $\mathbb{R}^4$ bestaat altijd uit 4 verschillende vectoren.
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- 1 W Als het systeem $A\vec{x} = \vec{b}$ meer dan 1 oplossing heeft, dan geldt dit ook voor het systeem $A\vec{x} = \vec{0}$.
 - 2 V De kolomruimte van een 3×5 matrix is altijd 5-dimensionaal.
 - 3 W Als een vierkante matrix 0 als eigenwaarde heeft, is hij niet inverseerbaar.
 - 4 W Een basis voor $\mathbb{R}^3$ bestaat altijd uit 3 verschillende vectoren.

- 1 W Als het systeem $A\vec{x} = \vec{b}$ juist 1 oplossing heeft, dan heeft het systeem $A\vec{x} = \vec{0}$ enkel de nuloplossing.
- 2 V Een 4×4 matrix waarvan de kolomruimte 4-dimensionaal is, heeft determinant 0.
- 3 W Stel A is een 3×3 matrix met twee verschillende eigenwaarden, namelijk 1 en 2. Dan is zijn determinant 2 of 4.
- 4 W Als $\det(A^2) = 1$, dan is $\det(A) \in \{-1, 1\}$.

